Neighborhood Semantics for Modal Logic
Lecture 1

Eric Pacuit

University of Maryland, College Park
pacuit.org
epacuit@umd.edu

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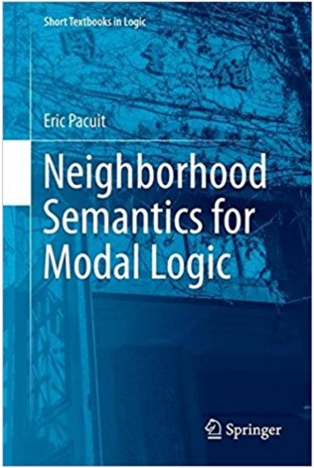
Text


Short Textbooks in Logic
Eric Pacuit
Neighborhood Semantics for Modal Logic
Springer

pacuit.org/modal/neighborhoods

Text

Ch 1: Introduction and Motivation

Ch 2: Core Theory: Expressivity, Completeness, Decidability, Complexity, Correspondence Theory

Ch 3: Richer Languages: Fixed-point operators, First-order extensions, Dynamic operators

Prerequisites

There are no specific prerequisites, although some background on modal logic will be helpful.

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Appendix A: Relational Models for Modal Logic

Schedule

Lecture 1: November 7, 15:30 - 17:30

Lecture 2: November 9, 14:00 - 16:00

Lecture 3: November 14, 15:30 - 17:30

1. Non-normal modal logics
2. Neighborhood semantics for modal logic

Normal modal logic

The Basic Modal Language: $\mathcal{L}$

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p is an atomic proposition (Let At be the set of atomic propositions)

One Language, Many Interpretations

tense: henceforth, eventually, previously, now, tomorrow, yesterday, since, until, it will have been, it is being,. . .

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geometric: it is locally the case that

metalogic: it is valid/satisfiable/provable/consistent that

game/action: there exist a strategy/action to guarantee that

Relational Structures

Relational (Kripke) Frame: $\langle W, R\rangle$

- $W \neq \emptyset$
- $R \subseteq W \times W$

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- $W \neq \emptyset$
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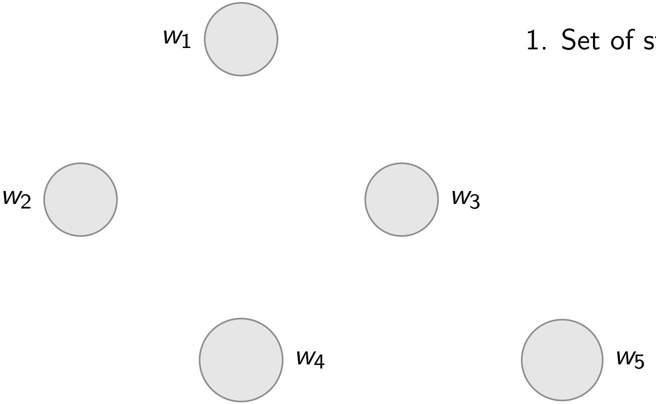
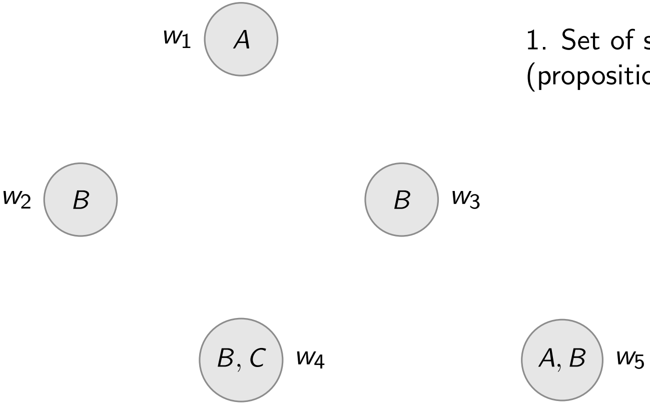
Relational (Kripke) Model: $\langle W, R, V \rangle$

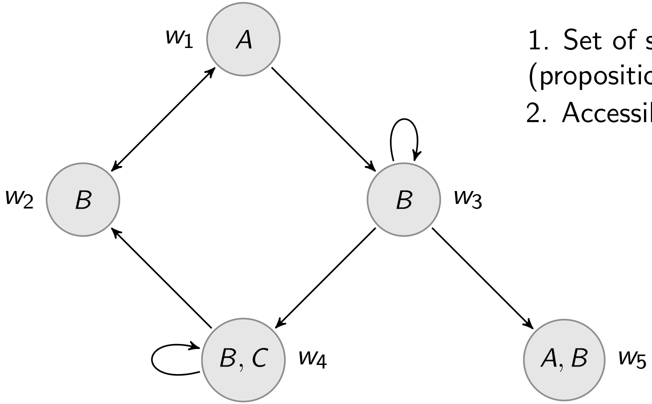
- $\langle W, R \rangle$ is a frame
- $V : \mathsf{At} \to \wp(W)$



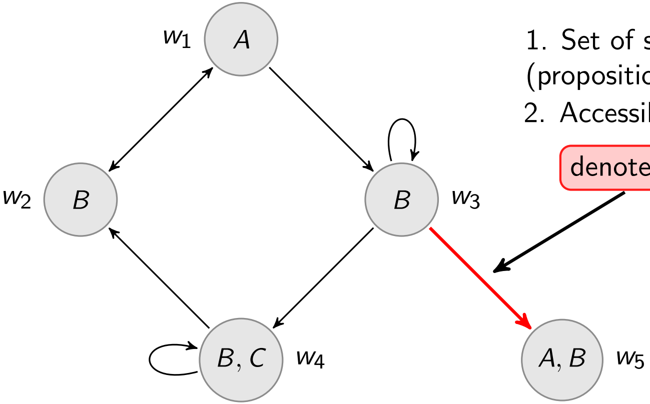
1. Set of states



1. Set of states (propositional valuations)



1. Set of states (propositional valuations)
2. Accessibility relation



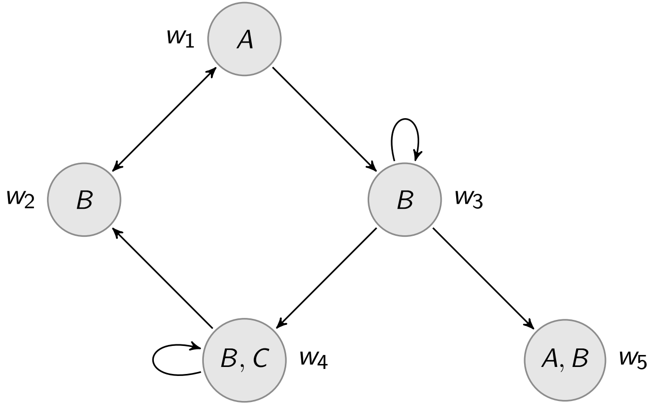
1. Set of states (propositional valuations)
2. Accessibility relation

denoted $w_3 R w_5$

Truth: $\mathcal{M}, w \models \varphi$

1. $\mathcal{M}, w \models p$ iff $w \in V(p)$
2. $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
3. $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
4. $\mathcal{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathcal{M}, v \models \varphi$
5. $\mathcal{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \models \varphi$

Example



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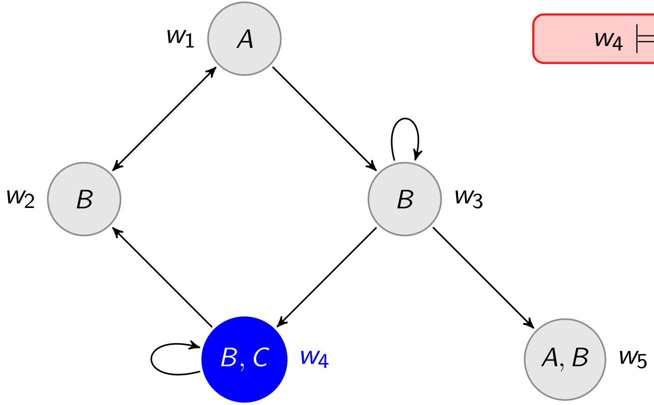
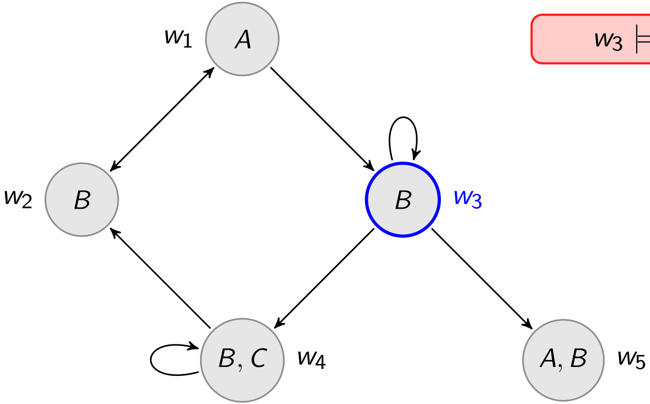


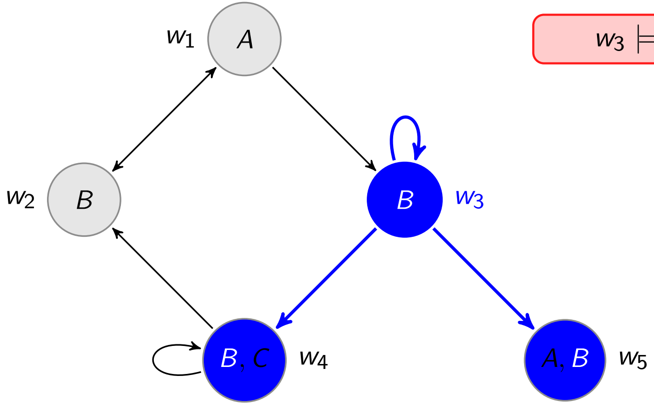
$$w_4 \models B \wedge C$$

Example



$w_3 \models \Box B$

Example



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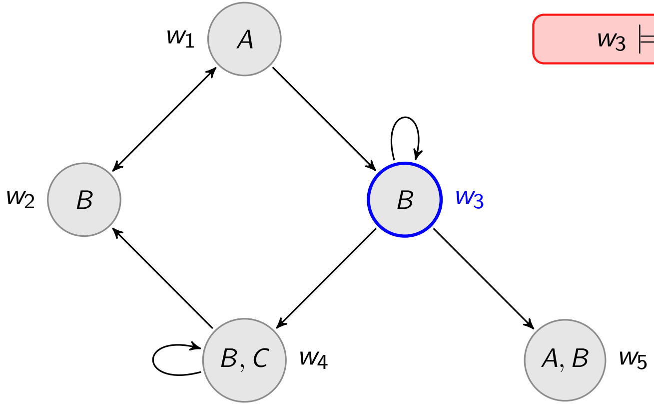
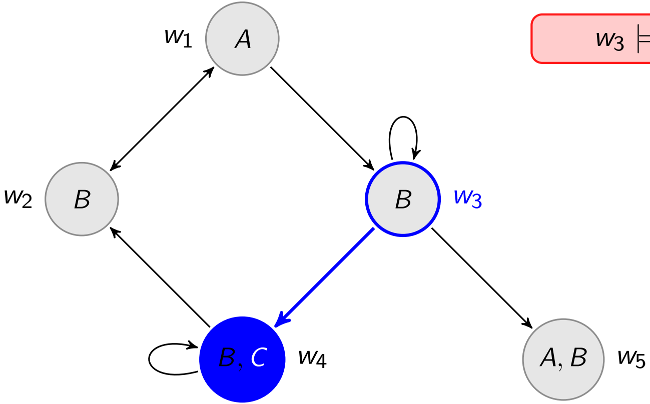
Example



$w_3 \models \Diamond C$

Example



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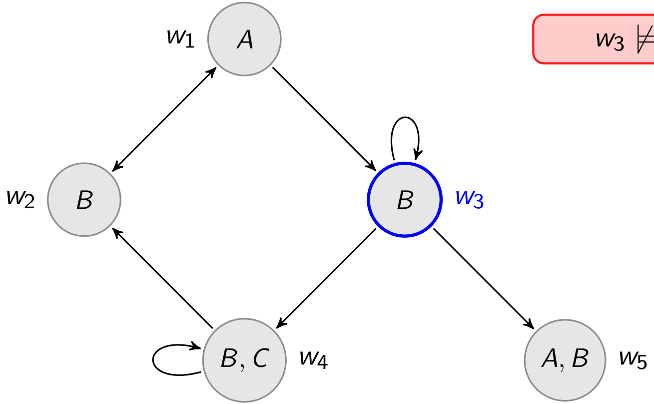
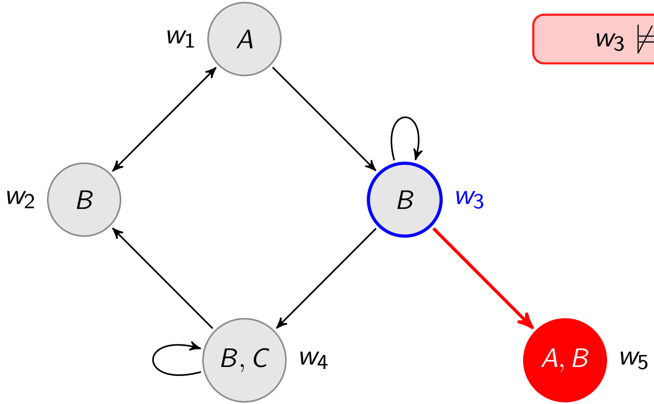


$w_3 \not\models \Box C$

Example



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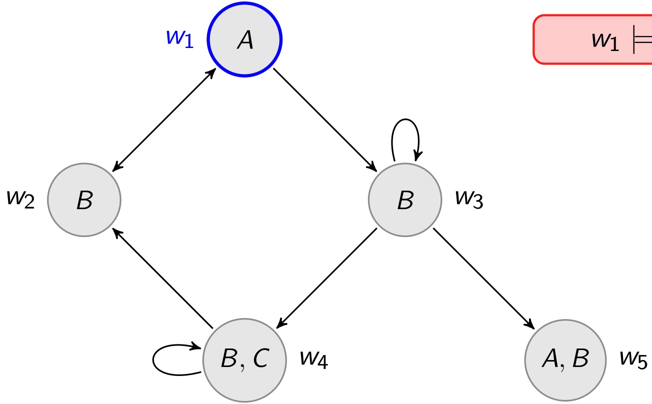
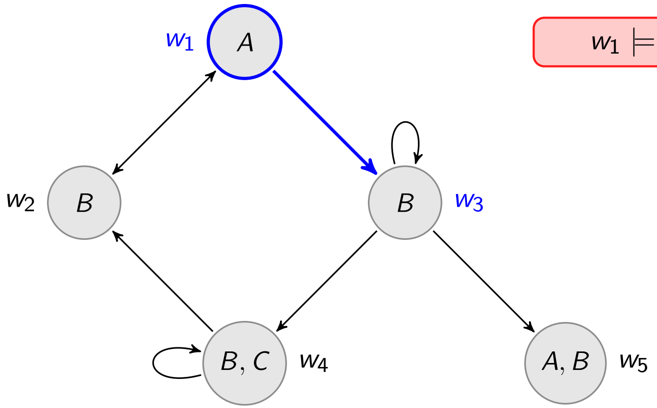


$w_1 \models \Diamond\Box B$

Example



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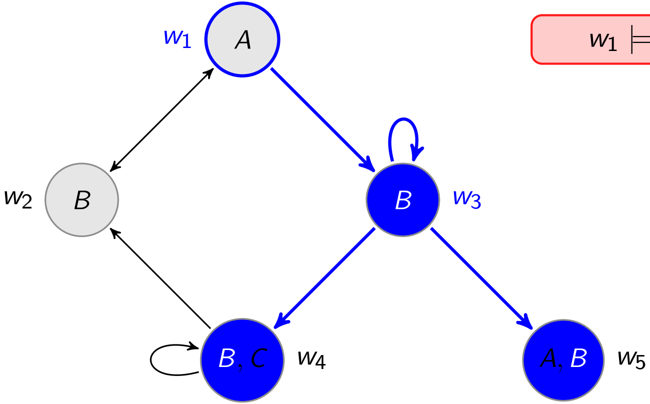
Example



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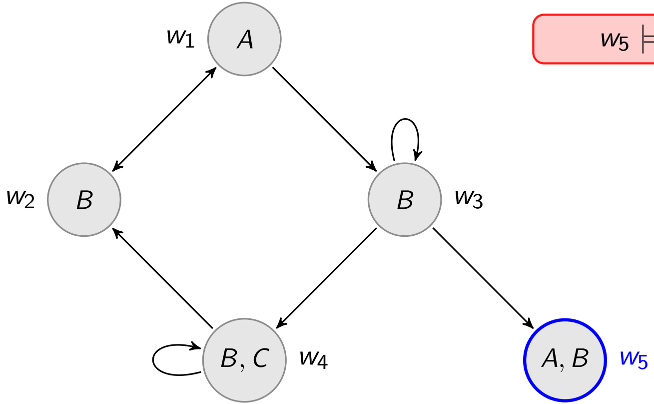
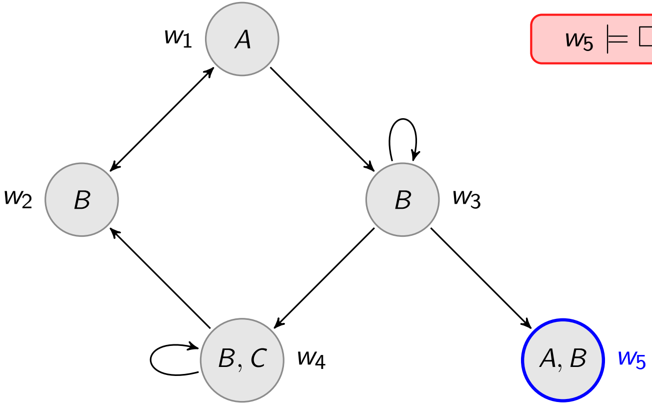


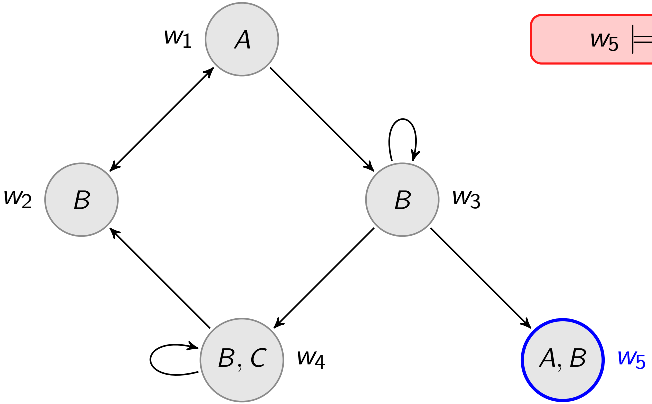
$w_5 \models \Box C$

Example



$w_5 \models \Box(B \wedge \neg B)$

Example



$w_5 \models \neg\Diamond B$

Standard Logical Notions

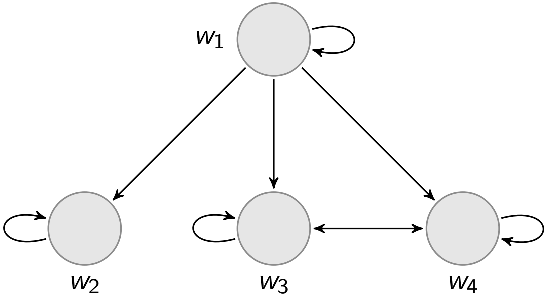
Valid on a model: $\mathcal{M} \models \varphi$

Valid at a state on a frame: $\mathcal{F}, w \models \varphi$

Valid on a frame: $\mathcal{F} \models \varphi$

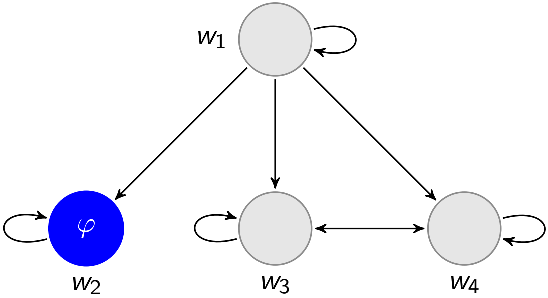
Valid in a class F of frame: $\models_{\mathsf{F}} \varphi$

Example



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \to \Diamond\Box\varphi$$

Example



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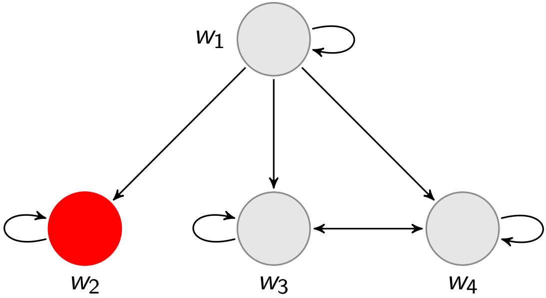

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Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

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(RM) $\dfrac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$

The History of Modal Logic

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, Vol. 7, 2006.

P. Balckburn, M. de Rijke, and Y. Venema. *Modal Logic*. Section 1.7, Cambridge University Press, 2001.

R. Ballarin. *Modern Origins of Modal Logic*. Stanford Encyclopedia of Philosophy, 2010.

Non-normal modal logics

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(Non-)Normal Modal Logic

Let $\mathcal{L}$ be the basic modal language.

A **modal logic** is a set of formulas from $\mathcal{L}$. If **L** is a modal logic, then we write $\vdash_{\mathbf{L}} \varphi$ when $\varphi \in \mathbf{L}$.

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A modal logic $\mathbf{L}$ is **normal** provided $\mathbf{L}$ is

- contains propositional logic (i.e., all instances of the propositional axioms and closed under Modus Ponens)
- closed under Necessitation (from $\vdash_{\mathbf{L}} \varphi$ infer $\vdash_{\mathbf{L}} \Box\varphi$);
- contains all instances of K ($\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$); and
- closed under *uniform substitution*.

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- closed under *uniform substitution*.

Normal Modal Logic

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

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Theorem. **K** is sound and strongly complete with respect to the class of all relational frames.

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Theorem. For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

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Theorem. **K** $+ \Box\varphi \to \varphi + \Box\varphi \to \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive relational frames.

Are there non-normal extensions of **K**?

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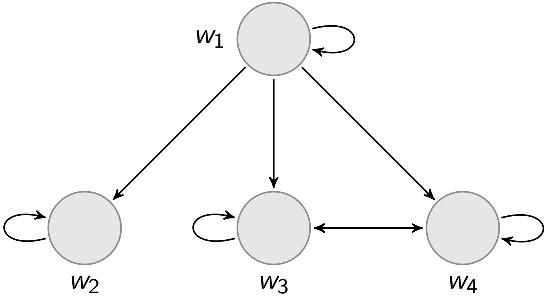
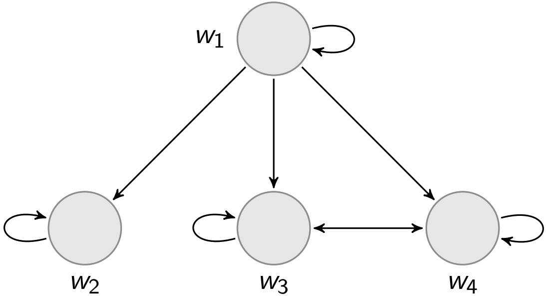
Let **L** be the smallest modal logic containing

- **S4** (**K** $+\ \Box\varphi \rightarrow \varphi\ +\ \Box\varphi \rightarrow \Box\Box\varphi$)
- all instances of M: $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.

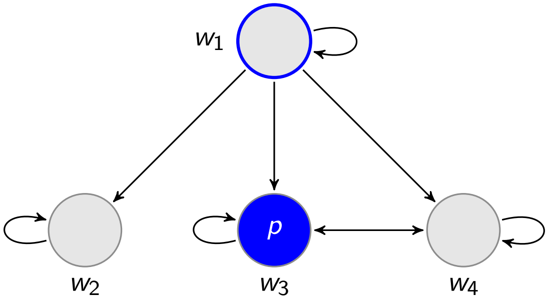


$\mathcal{F}, w_1 \models \Box\Diamond\varphi \to \Diamond\Box\varphi$



$$\mathcal{F}, w_1 \models \Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$$

$$\mathbf{L} \subseteq \mathbf{L}_{w_1} = \{\varphi \mid \mathcal{F}, w_1 \models \varphi\}$$



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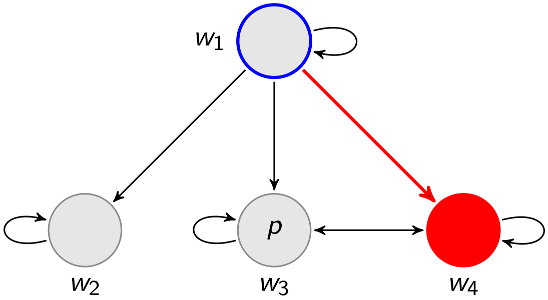
$\mathcal{F}, w_1 \not\models \Box(\Box\Diamond p \rightarrow \Diamond\Box p)$



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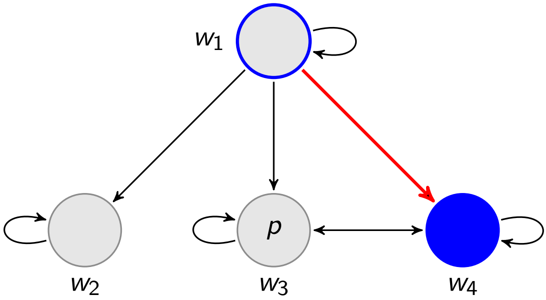

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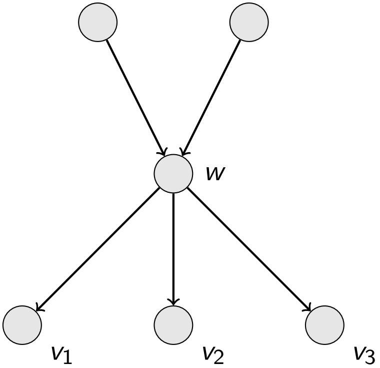
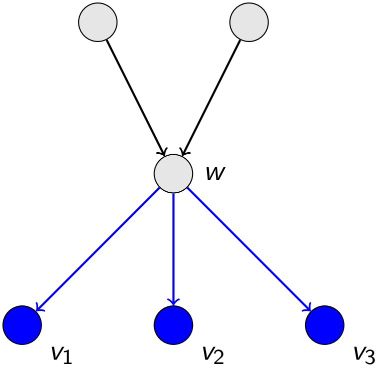
✓ Non-normal modal logics

1. Neighborhood semantics for modal logic


w
v_1
v_2
v_3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v1
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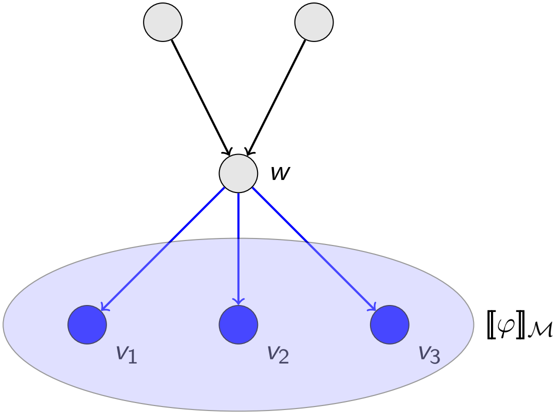
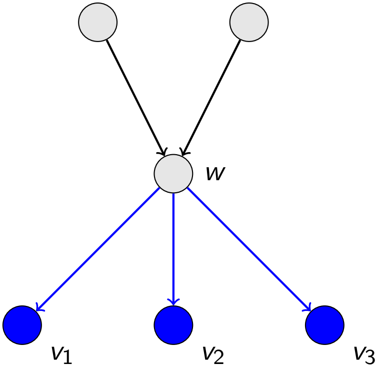
v3



$\mathcal{M}, w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M}}$

...**the** neighborhood of w is **contained in** the truth-set of φ



$\mathcal{M}, w \models \boxplus\varphi$ iff $R(w) = [\![\varphi]\!]_{\mathcal{M}}$

...**the** neighborhood of w **is** the truth-set of φ

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

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What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

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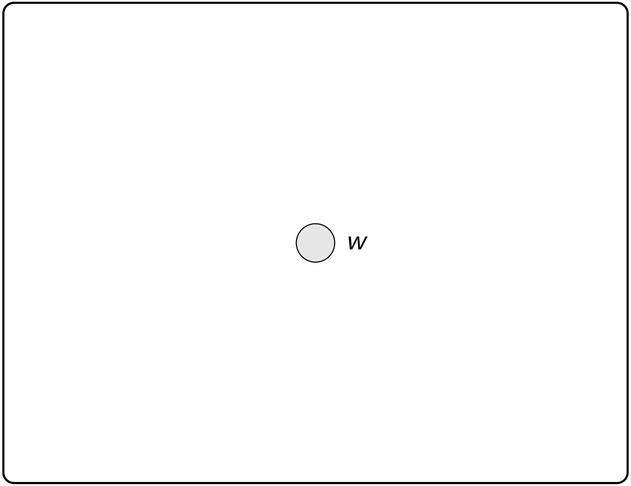
S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.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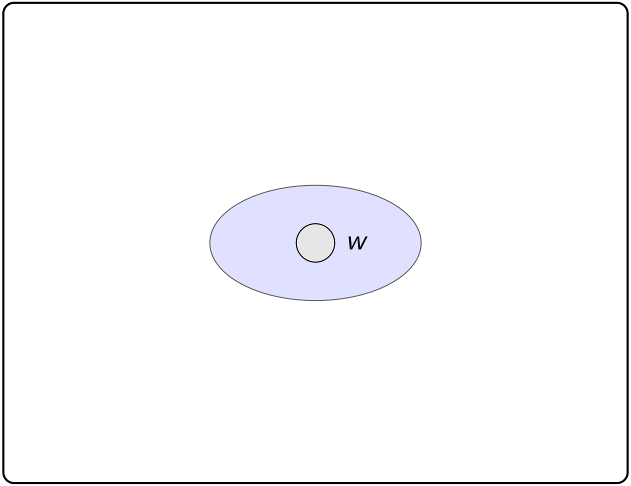
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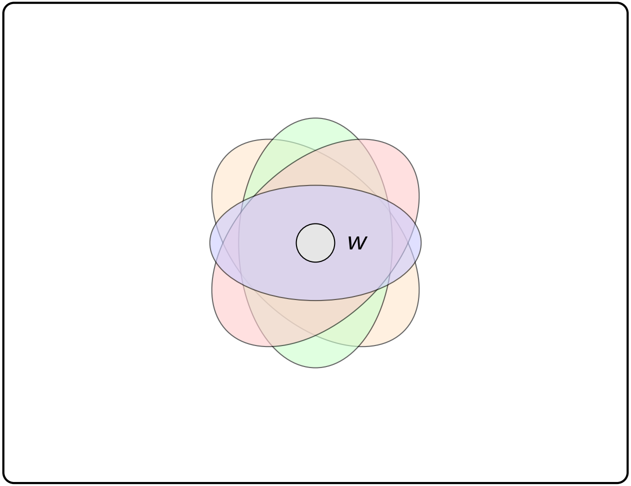
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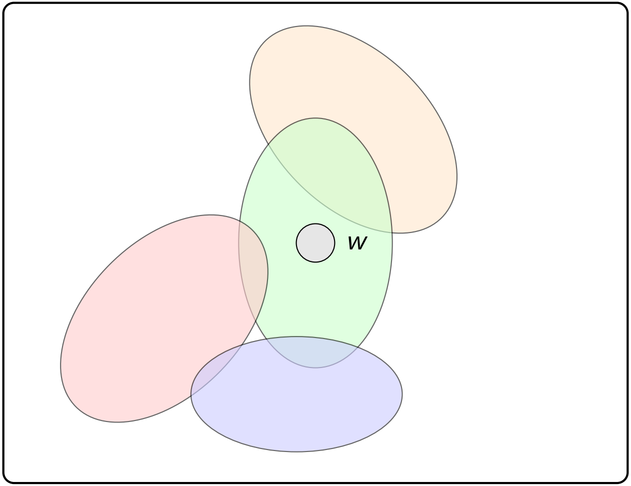
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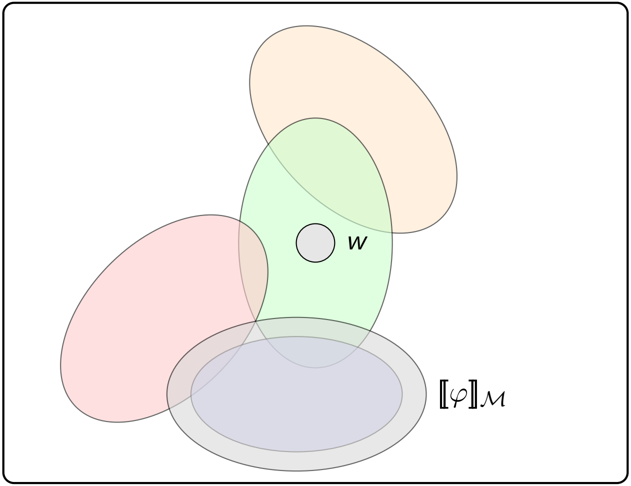

$\mathcal{M}, w \models \Box\varphi$ iff **there is** a neighborhood of w **contained in** $[\![\varphi]\!]_{\mathcal{M}}$

Relational model: $\langle W, R, V \rangle$ where $R : W \to \wp(W)$

$w \models \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]$

Neighborhood model: $\langle W, N, V \rangle$ where $N : W \to \wp(\wp(W))$

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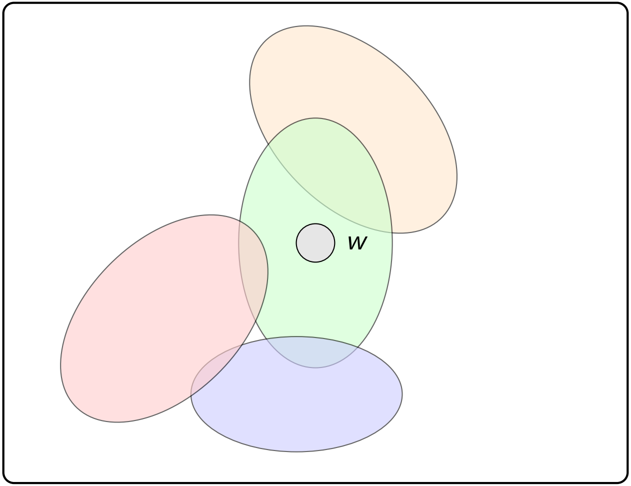

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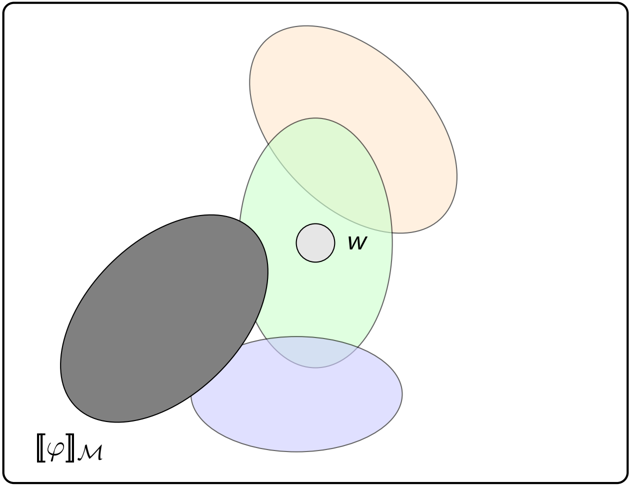

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✓ Non-normal modal logics

✓ Neighborhood semantics for modal logic

Why non-normal modal logic?

Why neighborhood models?

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense.... Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.

(pg. 1)

Two routes to a logical framework

1. Identify interesting patterns that you (do not) want to represent
2. Identify interesting structures that you want to reason about

- Logical omniscience
- Logics of knowledge and beliefs
- Logic of high probability
- Logics of ability
- Deontic logics
- Logics of classical deduction
- Logics of group decision making

Logical Omniscience/Knowledge Closure

RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

Nec From φ, infer $\Box\varphi$

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

Logical Omniscience/Knowledge Closure

RM From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$
closure under logical implication

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RM From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$
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knowledge of all logical validities

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$

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Nec From φ, infer $\Box\varphi$
knowledge of all logical validities

RE From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$
closure under logical equivalence

Logical Omniscience/Knowledge Closure

RM	From $\varphi \rightarrow \psi$, infer $\Box\varphi \rightarrow \Box\psi$ *closure under logical implication*
K	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *closure under known implication*
Nec	From φ, infer $\Box\varphi$ *knowledge of all logical validities*
RE	From $\varphi \leftrightarrow \psi$, infer $\Box\varphi \leftrightarrow \Box\psi$ *closure under logical equivalence*

Logical Omniscience/Knowledge Closure

W. Holliday. *Epistemic closure and epistemic logic I: Relevant alternatives and subjunctivism*. Journal of Philosophical Logic, 1 - 62, 2014.

J. Halpern and R. Puccella. *Dealing with logical omniscience: Expressiveness and pragmatics*. Artificial Intelligence 175(1), pgs. 220 - 235, 2011.

Logics of High Probability

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

R. Stalnaker. *On logics of knowledge and belief*. Philosophical Studies 128, 169 199, 2006.

(K)	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$
(T)	$K\varphi \to \varphi$
(4)	$K\varphi \to KK\varphi$
(Nec)	From φ infer $K\varphi$

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(4)	$K\varphi \to KK\varphi$
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(PI)	$B\varphi \to KB\varphi$
(NI)	$\neg B\varphi \to K\neg B\varphi$
(KB)	$K\varphi \to B\varphi$
(D)	$B\varphi \to \langle B\rangle\varphi$
(SB)	$B\varphi \to BK\varphi$

$$(.2) \qquad \langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi$$

$$(\text{DefKB}) \qquad B\varphi \leftrightarrow \langle K\rangle K\varphi$$

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What happens if we drop axiom (4)?

$$(.2) \qquad \langle K\rangle K\varphi \rightarrow K\langle K\rangle\varphi$$

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Claim. B is a normal modal operator.

What happens if we drop axiom (4)?

Under certain conditions, B is not a normal modal operator.

D. Klein, N. Gratzl, and O. Roy. *Introspection, normality and agglomeration*. Logic, Rationality, and Interaction, 5th Workshop, LORI 2015, 195 206.

Logic of Deduction

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Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the set of propositional formulas.

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Interpretation: $(\cdot)^* : \mathsf{At} \to \wp(\Sigma)$

- $(\varphi \vee \psi)^* = (\varphi)^* \cup (\psi)^*$
- $(\neg\varphi)^* = \Sigma - (\varphi)^*$
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Fact: $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ is not valid.

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Validities: $\varphi \to \Box\varphi$, (Mon), $\Box(\varphi \vee \Box\varphi) \to \Box\varphi$

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(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

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Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

$$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

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1. Jones murders Smith
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Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

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$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

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(Mon) If Jones ought to murder Smith gently, then Jones ought to murder Smith

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✓ Jones ought to murder Smith gently

5. If Jones murders Smith gently, then Jones murders Smith.

✓ If Jones ought to murder Smith gently, then Jones ought to murder Smith

7. Jones ought to murder Smith

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4. Jones ought to murder Smith gently
5. If Jones murders Smith gently, then Jones murders Smith.
6. If Jones ought to murder Smith gently, then Jones ought to murder Smith
- ✗ Jones ought to murder Smith

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

Why non-normal modal logic? ✓

Why neighborhood models?

- Subset spaces, neighborhood frames/models, reasoning about subset spaces
- Interesting mathematical structures: Ultrafilters, topologies, hypergraphs
- Logic of knowledge, evidence and belief
- Coalitional logic

Some Terminology: Subset Spaces

Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under unions if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under complements if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

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- the set $\cap_{X\in\mathcal{F}}X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X\in\mathcal{F}}X \in \mathcal{F}$.
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- $\mathcal{F}$ is normal if $\mathcal{F} \neq \emptyset$.

A few more definitions

- $\mathcal{F}$ is a filter if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.

- $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.

- $\mathcal{F}$ is a topology if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.

- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $[\![p]\!]_{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $[\![\neg\varphi]\!]_{\mathfrak{M}} = W - [\![\varphi]\!]_{\mathfrak{M}}$
3. $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} = [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$
4. $[\![\Box\varphi]\!]_{\mathfrak{M}} = m_N([\![\varphi]\!]_{\mathfrak{M}})$
5. $[\![\Diamond\varphi]\!]_{\mathfrak{M}} = W - m_N(W - [\![\varphi]\!]_{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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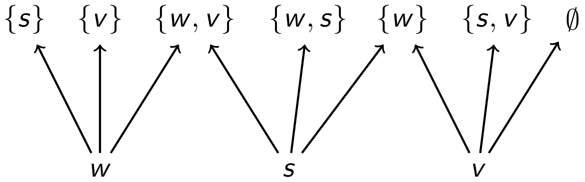

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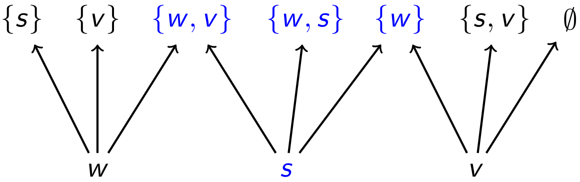

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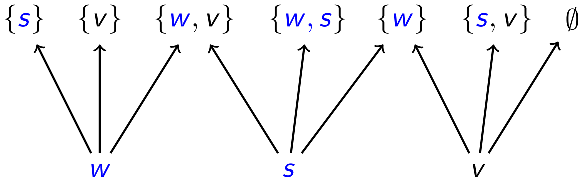

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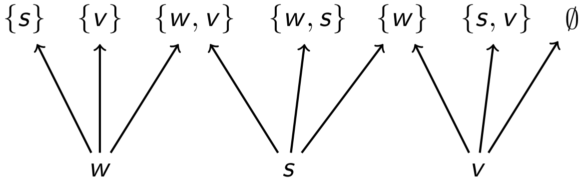

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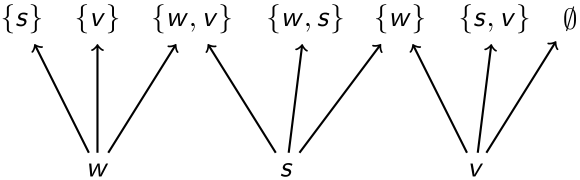

$$\mathfrak{M}, s \models \Box p$$

Detailed Example

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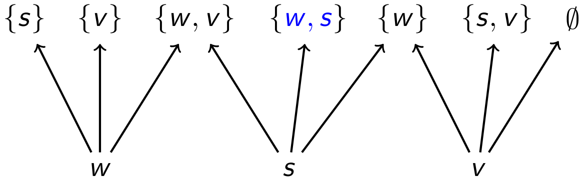

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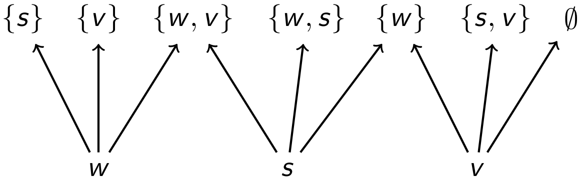
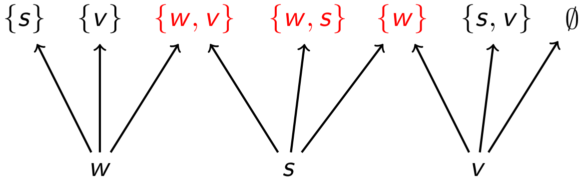


$$\mathfrak{M}, s \models \Diamond p$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$

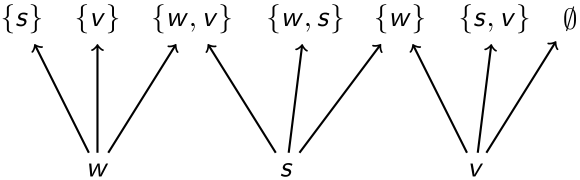


$$\mathfrak{M}, s \models \Diamond p$$

$$[\![\neg p]\!]_{\mathfrak{M}} = \{v\}$$

Detailed Example

$V(p) = \{w, s\}$ and $V(q) = \{s, v\}$



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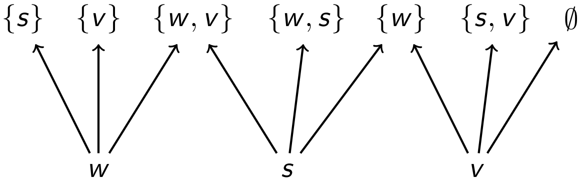

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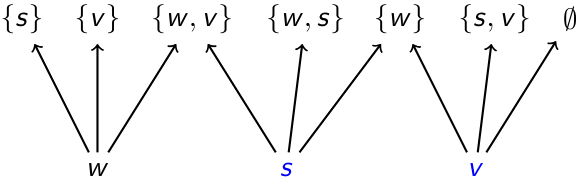

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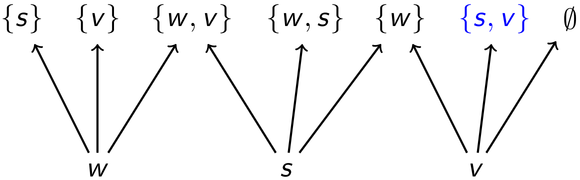

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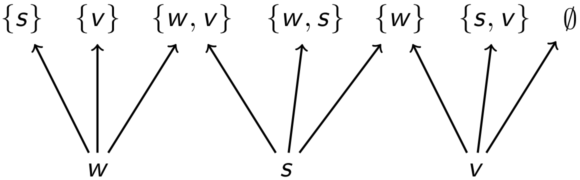

$\mathfrak{M}, w \not\models \Diamond\Box p$

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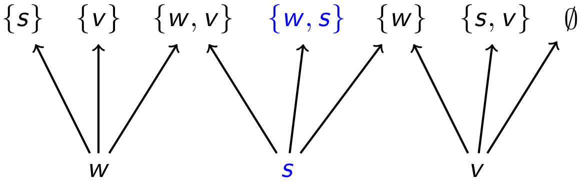

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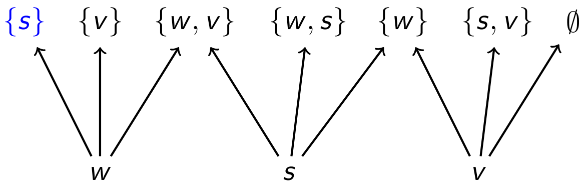

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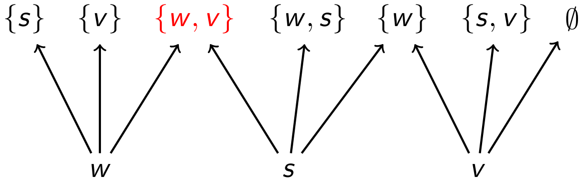

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$\mathfrak{M}, w \models \Box\Box p$

$\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, v \models \Diamond\Box p$

Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
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Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Lemma

1. *If* $\varphi \to \psi$ *is valid in* $\mathfrak{M}$*, then so is* $\langle\ ]\varphi \to \langle\ ]\psi$.
2. $\langle\ ](\varphi \wedge \psi) \to (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid in* $\mathfrak{M}$

Investigate analogous results for the other modal operators defined above.

Non-normal modal logics

(M)	$\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
(C)	$\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
(N)	$\Box\top$
(K)	$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(Dual)	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
(Nec)	from $\vdash \varphi$ infer $\vdash \Box\varphi$
(Re)	from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Non-normal modal logics

~~(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$~~

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~~(N) $\Box\top$~~

~~(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$~~

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

~~(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$~~

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad\quad \varphi \rightarrow \psi}{\psi}$

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Nec $\dfrac{\varphi}{\Box\varphi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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EM is the logic **E** + *Mon*

PC 6. Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

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N $\Box\top$

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EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

PC Propositional Calculus

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EMC is the smallest regular modal logic

PC Propositional Calculus

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Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

Mon $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

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MP $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

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EMC is the smallest regular modal logic

K is the smallest normal modal logic

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

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N $\Box\top$

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RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = **EMCN**

PC Propositional Calculus

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Mon $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

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RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

EC is the logic **E** + *C*

EMC is the smallest regular modal logic

K = *PC*(+*E*) + *K* + *Nec* + *MP*

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Why non-normal modal logic?

Why neighborhood models?

Abilities

$Abl_i\varphi$: i has the ability to *see to it that* φ is true
(alternatively, i has the ability to bring about φ)

What are the core logical principles?

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1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)

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2. $\neg Abl_i\top$

Abilities

$Abl_i\varphi$: i has the ability to *see to it that* φ is true
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What are the core logical principles?

1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$

Abilities

$Abl_i\varphi$: i has the ability to *see to it that* φ is true (alternatively, i has the ability to bring about φ)

What are the core logical principles?

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4. $Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$

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5. $Abl_i(\varphi \wedge \psi) \rightarrow (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \rightarrow Abl_i\varphi$, $Abl_iAbl_i\varphi \rightarrow Abl_i\varphi$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

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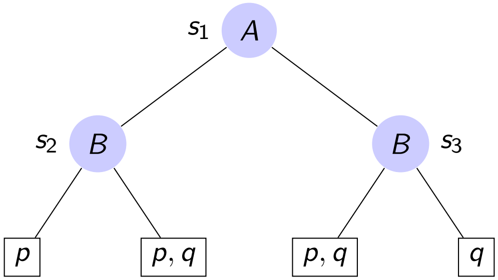
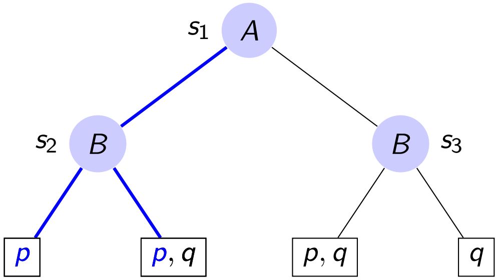
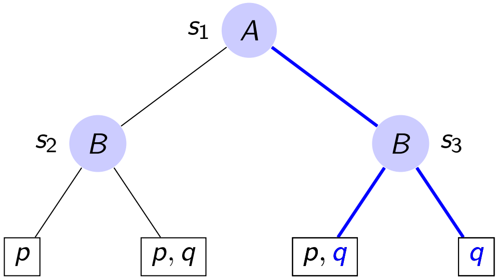


Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



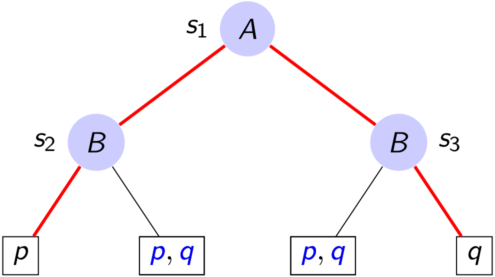
$$s_1 \models Abl_A p$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$



$$s_1 \models Abl_A p \wedge Abl_A q \wedge \neg Abl_A(p \wedge q)$$

Games: $(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

R. Parikh. *The Logic of Games and its Applications*. Annals of Discrete Mathematics (1985).

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica (2003).

J. van Benthem. *Logic and Games*. Course notes (2007).

Question

$\Box_i \varphi$ means "player i has a strategy to win the game"

$\Diamond_i \varphi$ means "player i's opponent has a strategy to win the game"

Question

$\Box_i\varphi$ means "player i has a strategy to win the game"

$\Diamond_i\varphi$ means "player i's opponent has a strategy to win the game"

- Is $\neg\Diamond_i\neg\varphi \rightarrow \Box_i\varphi$ valid?
- Is $\Box_i\varphi \rightarrow \neg\Diamond_i\neg\varphi$ valid? Hint: the formula is equivalent to $\neg(\Box_i\varphi \wedge \Diamond_i\neg\varphi)$

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

$\varphi \not\to Abl_i\varphi$

Suppose an agent (call her Ann) is throwing a dart and she is not a very good dart player, but she just happens to throw a bull's eye.

Intuitively, we do not want to say that Ann has the *ability* to throw a bull's eye even though it happens to be true.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\to Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

$Abl_i(\varphi \vee \psi) \not\rightarrow Abl_i\varphi \vee Abl_i\psi$

Continuing with this example, suppose that Ann has the ability to hit the dart board, but has no other control over the placement of the dart.

Now, when she throws the dart, as a matter of fact, it will either hit the top half of the board or the bottom half of the board.

Since, Ann has the ability to hit the dart board, she has the ability to either hit the top half of the board or the bottom half of the board.

However, intuitively, it seems true that Ann does not have the ability to hit the top half of the dart board, and also she does not have the ability to hit the bottom half of the dart board.

Abilities

$Abl_i\varphi$: agent i has the ability to bring about (see to it that) φ is true

What are core logical principles? Depends very much on the intended "application" and how actions are represented...

1. $Abl_i\varphi \to \varphi$ (or $\varphi \to Abl_i\varphi$)
2. $\neg Abl_i\top$
3. $(Abl_i\varphi \wedge Abl_i\psi) \to Abl_i(\varphi \wedge \psi)$
4. $Abl_i(\varphi \vee \psi) \to (Abl_i\varphi \vee Abl_i\psi)$
5. $Abl_i(\varphi \wedge \psi) \to (Abl_i\varphi \wedge Abl_i\psi)$
6. $Abl_iAbl_j\varphi \to Abl_i\varphi$, $Abl_iAbl_i\varphi \to Abl_i\varphi$

On the Logic of Ability

$Abl_i\top$

$\varphi \rightarrow Abl_i\varphi$

$(Abl_i\varphi \wedge Abl_i\psi) \rightarrow Abl_i(\varphi \wedge \psi)$

$Abl_i(\varphi \vee \psi) \rightarrow (Abl_i\varphi \vee Abl_i\psi)$

On the Logic of Ability

$\neg Abl_i \top$

$\varphi \not\to Abl_i \varphi$

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On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

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On the Logic of Ability

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$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i \varphi$

$\varphi \to \Diamond\varphi$ is valid in the class of reflexive frames

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$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

On the Logic of Ability

$\neg Abl_i \top$

$\Box\top$ is valid in the class of all frames, $\Diamond\top$ is valid on the class of serial frames

$\varphi \not\to Abl_i\varphi$

$\varphi \to \Diamond\varphi$ is valid in the class of reflexive frames

$(Abl_i\varphi \wedge Abl_i\psi) \not\to Abl_i(\varphi \wedge \psi)$

$(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ is valid in the class of all frames

$Abl_i(\varphi \vee \psi) \not\to (Abl_i\varphi \vee Abl_i\psi)$

$\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$ is valid in the class of all frames

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"

(p. 135)

A. Kenny. *Will, Freedom and Power*. 1975.

Ability: Reproducibility vs. Reliability

"Abilities are inherently general; there are no genuine abilities which are abilities to do things only on one particular occasion"
(p. 135)

A. Kenny. *Will, Freedom and Power*. 1975.

"Even if opportunity only knocks once, I may be able to act on it, and may be culpable for doing so, or for failing to do so."
(p. 1)

M. Brown. *On the Logic of Ability*. Journal of Philosophical Logic, Vol. 17, pp. 1 - 26, 1988.

D. Elgesem. *The modal logic of agency*. Nordic Journal of Philosophical Logic 2(2), 1 - 46, 1997.

G. Governatori and A. Rotolo. *On the Axiomatisation of Elgesem's Logic of Agency and Ability*. Journal of Philosophical Logic, 34, pgs. 403 - 431 (2005).

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

A Minimal Logic of Abilities

$C\varphi$ means "the agent is capable of realizing φ"

$E\varphi$ means "the agent does bring about φ"

1. All propositional tautologies
2. $\neg C\top$
3. $E\varphi \wedge E\psi \rightarrow E(\varphi \wedge \psi)$
4. $E\varphi \rightarrow \varphi$
5. $E\varphi \rightarrow C\varphi$
6. Modus Ponens plus from $\varphi \leftrightarrow \psi$ infer $E\varphi \leftrightarrow E\psi$ and from $\varphi \leftrightarrow \psi$ infer $C\varphi \leftrightarrow C\psi$

Problems

- Appendix A: Exercises 89 & 91
- Sections 1.1 & 1.2: Exercises 1 - 7
- Section 2.3: Exercises 39 - 43