Neighborhood Semantics for Modal Logic

Lecture 3

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Schedule

- ✓ Lecture 1: November 7, 15:30 - 17:30
- ✓ Lecture 2: November 9, 14:00 - 16:00
- Lecture 3: November 14, 15:30 - 17:30

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

MP $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

(*Mon*) $\dfrac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$

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EM is the logic **E** $+$ *Mon*

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EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

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K $=$ **EMCN**

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EMC is the smallest regular modal logic

K $=$ *PC*$(+E)$ + *K* + *Nec* + *MP*

Neighborhood Frames

Let W be a non-empty set of states.

Any function $N : W \to \wp(\wp(W))$ is called a neighborhood function

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

A neighborhood model based on $\mathfrak{F} = \langle W, N \rangle$ is a tuple $\langle W, N, V \rangle$ where $V : \text{At} \to \wp(W)$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $[\![\varphi]\!]_{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - [\![\varphi]\!]_{\mathfrak{M}} \notin N(w)$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

The Broader Logical Landscape

- Relational Models
- Topological Models
- *n*-ary Relational Structures
- Plausibility Structures
- First-Order Logic

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- Bisimulation
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- **EC** $\nvdash K$
- **EM** $\nvdash K$
- **EMC** $\vdash K$
- **EK** $\nvdash M$
- **EK** $\nvdash C$

Expressive Power and Invariance

M. Pauly. *Bisimulation for Non-normal Modal Logic*. Manuscript, 1999.

H. Hansen. *Monotonic Modal Logic*. ILLC, Masters Thesis, 2003.

Monotonic Bisimulation

Suppose that $\mathfrak{M} = \langle W, N, V \rangle$ and $\mathfrak{M}' = \langle W', N', V' \rangle$ are two monotonic neighborhood models. A relation $Z \subseteq W \times W'$ is a **monotonic bisimulation** provided that, whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$.

Zig: If $w\, N\, X$ then there is an $X' \subseteq W'$ such that $w'\; N'\; X'$ and $\forall x' \in X'$, $\exists x \in X$ such that $x\, Z\, x'$.

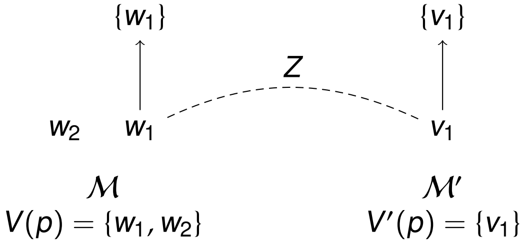
Zag: If $w'\; N'\; X'$ then there is an $X \subseteq W$ such that $w\, N\, X$ and $\forall x \in X$, $\exists x' \in X'$ such that $x\, Z\, x'$.

Write $\mathfrak{M}, w \underline{\leftrightarrow} \mathfrak{M}', w'$ when there is a monotonic bisimulation $Z \subseteq dom(\mathcal{M}) \times dom(\mathcal{M}')$ such that $w\, Z\, w'$.

Proposition. If $\mathcal{M}$ is a monotonic model, $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$ implies $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$.

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are monotonic, locally core-finite models. Then, for all $w \in W$, $w' \in W'$, $\mathcal{M}, w \equiv_{\mathcal{L}} \mathcal{M}', w'$ iff $\mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Do monotonic bisimulations work when we drop monotonicity?
No!


{w1}
{v1}
Z
w2
w1
v1
M
V(p) = {w1, w2}
M′
V′(p) = {v1}

Bounded Morphisms

If $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ are two neighborhood models, and $f : W_1 \to W_2$ is a function, then f is a **(frame) bounded morphism** if

for all $X \subseteq W_2$, we have $f^{-1}[X] \in N_1(w)$ iff $X \in N_2(f(w))$;

and for all $p \in \text{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

Bounded Morphisms

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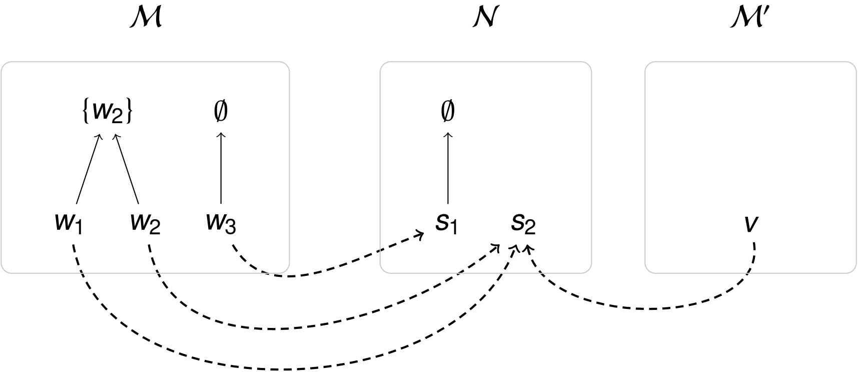
and for all $p \in \text{At}$, and all $w \in W_1$: $w \in V_1(p)$ iff $f(s) \in V_2(p)$.

Lemma Let $\mathcal{M}_1 = \langle W_1, N_1, V_1 \rangle$ and $\mathcal{M}_2 = \langle W_2, N_2, V_2 \rangle$ be two neighborhood models and $f : \mathcal{M}_1 \to \mathcal{M}_2$ a bounded morphism. For each modal formula $\varphi \in \mathcal{L}$ and state $w \in W_1$, $\mathcal{M}_1, w \models \varphi$ iff $\mathcal{M}_2, f(w) \models \varphi$.

Behavioral Equivalence

Definition

Two points w_1 from $\mathfrak{M}_1$ and w_2 from $\mathfrak{M}_2$ are **behaviorally equivalent** provided there is a neighborhood frame $\mathfrak{F}$ and bounded morphisms $f : \mathfrak{F}_1 \to \mathfrak{F}$ and $g : \mathfrak{F}_2 \to \mathfrak{F}$ such that $f(w_1) = g(w_2)$.



$\mathcal{M}$
$\mathcal{N}$
$\mathcal{M}'$
$\{w_2\}$
$\emptyset$
$\emptyset$
w_1
w_2
w_3
s_1
s_2
v

Proposition. Suppose that $\mathcal{M} = \langle W, N, V \rangle$ and $\mathcal{M}' = \langle W', N', V' \rangle$ are two neighborhood models. If states $w \in W$ and $w' \in W'$ are behaviorally equivalent, then for all $\varphi \in \mathcal{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}', w' \models \varphi$.

Core Theory

- ✓ Neighborhood Semantics in the Broader Logical Landscape
- ✓ Bisimulations
- Completeness, Decidability, Complexity
- Incompleteness
- Relation with Relational Semantics
- Model Theory

Suppose that Γ is a set of formulas and F is a class of neighborhood frames. A formula $\varphi \in \mathcal{L}$ is a **semantic consequence** with respect to F of Γ, denoted $\Gamma \models_{\mathsf{F}} \varphi$, provided for each model $\mathcal{M} = \langle W, N, V \rangle$ based on a frame from F (i.e., $\langle W, N \rangle \in \mathsf{F}$), for each $w \in W$, if $\mathcal{M}, w \models \Gamma$, then $\mathcal{M}, w \models \varphi$.

Some Notation

- A formula $\varphi \in \mathcal{L}$ is valid in F ($\models_{\mathsf{F}} \varphi$) if for each $\mathbb{F} \in$ F, $\mathbb{F} \models \varphi$.
- We say that a logic **L** is sound with respect to F, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathsf{F}} \varphi$.
- A logic **L** is weakly complete with respect to a class of frames F, if $\models_{\mathsf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.
- A logic **L** is strongly complete with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.

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A set of formulas Γ is called a **maximally consistent set** provided Γ is a consistent set of formulas and for all formulas $\varphi \in \mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of **L**-maximally consistent sets of formulas.

The **L**-proof set of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Let **L** be a logic and $\varphi, \psi \in \mathcal{L}$. Then

1. $|\varphi \wedge \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\varphi|_{\mathbf{L}} = M_{\mathbf{L}} - |\varphi|_{\mathbf{L}}$
3. $|\varphi \vee \psi|_{\mathbf{L}} = |\varphi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. $|\varphi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \rightarrow \psi$
5. $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$ iff $\vdash_{\mathbf{L}} \varphi \leftrightarrow \psi$
6. For any maximally **L**-consistent set Γ, if $\varphi \in \Gamma$ and $\varphi \rightarrow \psi \in \Gamma$, then $\psi \in \Gamma$
7. For any maximally **L**-consistent set Γ, If $\vdash_{\mathbf{L}} \varphi$, then $\varphi \in \Gamma$

Lindenbaum's Lemma. For any consistent set of formulas Γ, there exists a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$.

Canonical Model

Definition

A neighborhood model $\mathbb{M} = \langle W, N, V \rangle$ is canonical for **L** provided

- $W = \{$ maximally **L**-consistent sets $\}$

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- for all $\varphi \in \mathcal{L}$ and $\Gamma \in W$, $|\varphi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\varphi \in \Gamma$
- for all $p \in \text{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathcal{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N_{\mathbf{L}}^{min}(\Gamma) = \{|\varphi|_{\mathbf{L}} \mid \Box\varphi \in \Gamma\}$.

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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathcal{M}_{\mathbf{L}}^{max} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{max}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N_{\mathbf{L}}^{max}(\Gamma) = N_{\mathbf{L}}^{min}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathcal{M}$ *is canonical for* $\mathbf{L}$,

$$[\![\varphi]\!]_{\mathcal{M}} = |\varphi|_{\mathbf{L}}$$

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The Proofs

Theorem

The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

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The logic **E** *is sound and strongly complete with respect to the class of all neighborhood frames.*

Lemma

If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem

The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}} \rangle$ is not closed under supersets.

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Lemma

Suppose that $\mathcal{M} = sup(\mathcal{M}^{min}_{\mathbf{EM}})$. *Then* $\mathcal{M}$ *is canonical for* **EM**.

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

The Normal Situation

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \rightarrow \psi \qquad \vdash \varphi}{\psi}$

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Theorem: K is sound and strongly complete with respect to the class of all Kripke frames.

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MP $\dfrac{\vdash \varphi \to \psi \qquad \vdash \varphi}{\psi}$

Theorem: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

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Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \rightarrow \psi \qquad \vdash \varphi}{\psi}$

Theorem: K $+ \Box\varphi \rightarrow \varphi + \Box\varphi \rightarrow \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive Kripke frames.

A logic **L** is **neighborhood complete** (resp. **Kripke complete**) provided there is a class of neighborhood frames F (resp. relational frames) such that **L** $= \mathbf{L}(\mathsf{F}) = \{\varphi \in \mathcal{L} \mid \mathbb{F} \models \varphi$ for all $\mathbb{F} \in \mathsf{F}\}$. Otherwise, the logic is said to be **neighborhood incomplete** (resp. **Kripke incomplete**).

Incompleteness

There are (consistent) modal logics that are incomplete:

Incompleteness

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Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \rightarrow \varphi$
- $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \rightarrow \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \rightarrow \Box\varphi)) \rightarrow \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

Incompleteness

There are (consistent) modal logics that are incomplete:

Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \to \varphi$
- $\Box\Diamond\varphi \to \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \to \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \to \Box\varphi)) \to \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

J. van Benthem. *Two Simple Incomplete Modal Logics*. Theoria (1978).

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames? No

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and **L**$'$ respectively, but φ and φ' are not deducible in the respective logics).

Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

There are two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L′ is above **S4** (adapts Fine's incomplete logic)

$$
\begin{array}{rcl}
A_i & = & \Box(q_i \to r) \qquad (i = 1,2) \\
B_i & = & \Box(r \to \Diamond q_i) \qquad (i = 1,2) \\
C_1 & = & \Box\neg(q_1 \wedge q_2) \\
A & = & r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge \\
 & & C_1 \to \Diamond(r \wedge \Box(r \to (q_1 \vee q_2)) \\
D & = & (p \wedge \Diamond\Diamond q) \to (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p)) \\
E & = & (\Box p \wedge \neg\Box\Box p) \to \Diamond(\Box\Box p \wedge \neg\Box\Box\Box p) \\
F & = & \Box p \to \Box\Box p
\end{array}
$$

$$
\begin{aligned}
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D &= (p \wedge \Diamond\Diamond q) \to (\Diamond q \vee \Diamond\Diamond(q \wedge \Diamond p)) \\
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\end{aligned}
$$

Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

$$
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A &= r \wedge \Box p \wedge \neg\Box\Box p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge\\
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F &= \Box p \to \Box\Box p
\end{aligned}
$$

Let **L** be the logic obtained by adding A, D, and E as additional axioms to **T**.

Theorem. (Gerson) The formula F is valid in all neighborhood frames for **L**, but it is not provable in **L**.

Comparing Relational and Neighborhood Semantics

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Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

Comparing Relational and Neighborhood Semantics

Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics? Yes!

Comparing Relational and Neighborhood Semantics

Neighborhood completeness does not imply Kripke completeness

- extension of **K**

D. Gabbay. *A normal logic that is complete for neighborhood frames but not for Kripke frames*. Theoria (1975).

- extension of **T**

M. Gerson. *A Neighbourhood frame for T with no equivalent relational frame*. Zeitschr. J. Math. Logik und Grundlagen (1976).

- extension of **S4**

M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

W. Holliday and T. Litak. *Complete Additivity and Modal Incompleteness*. The Review of Symbolic Logic.

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Summary

For any consistent modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of *general frames*

Summary

For any consistent modal logic **L**:

- If **L** is Kripke complete, then it is neighborhood complete
- **L** is complete with respect to its class of *general frames*

There are modal logics showing that

- neighborhood completeness does not imply Kripke completeness
- algebraic completeness does not imply neighborhood completeness

Frame Correspondence

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma
Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

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Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ *iff* $\mathfrak{F}$ *is closed under finite intersections.*

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

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Consider the formulas $\Diamond\top$ and $\Box\varphi \to \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \to \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma
Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \rightarrow \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
3. $\Diamond\varphi \rightarrow \Box\varphi$
4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

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4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

What about more general results (e.g., Sahlqvist Theorem, Standard Translation, van Benthem Characterization Theorem, etc.?)

We can *simulate* any non-normal modal logic with a bi-modal normal modal logic.

Definition

Given a neighborhood model $\mathcal{M} = \langle W, N, V \rangle$, define a Kripke model $\mathcal{M}^{\circ} = \langle V, R_N, R_{\not\ni}, R_N, Pt, V \rangle$ as follows:

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- $R_N = \{(w, u) \mid w \in W, u \in \wp(W), u \in N(w)\}$
- $Pt = W$

Let $\mathcal{L}'$ be the language

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid [\ni]\varphi \mid [\not\ni]\varphi \mid [N]\varphi \mid \mathrm{Pt}$$

where $p \in \mathrm{At}$ and Pt is a unary modal operator.

Define $ST : \mathcal{L} \rightarrow \mathcal{L}'$ as follows

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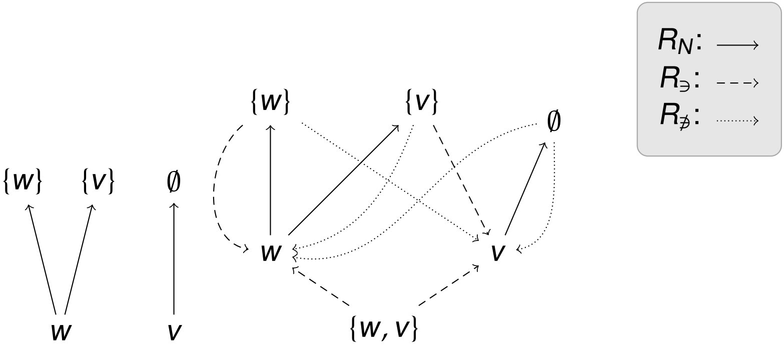
Lemma

For each neighborhood model $\mathcal{M} = \langle W, N, V \rangle$ and each formula $\varphi \in \mathcal{L}$, for any $w \in W$,

$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}^\circ, w \models ST(\varphi)$$



$\mathfrak{M}$ $\mathfrak{M}^\circ$

$\mathfrak{M}, w \models \Box p$ and $\mathfrak{M}, v \models \Box\bot$.

- $\mathfrak{M}^\circ, w \models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$ and $\mathfrak{M}^\circ, v \not\models \langle N\rangle([\ni]p \wedge [\not\ni]\neg p)$
- $\mathfrak{M}^\circ, v \models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$ and $\mathfrak{M}^\circ, w \not\models \langle N\rangle([\ni]\bot \wedge [\not\ni]\top)$

Monotonic Models

Lemma

On Monotonic Models $\langle N\rangle([\ni]ST(\varphi) \wedge [\not\ni]\neg ST(\varphi))$ *is equivalent to* $\langle N\rangle([\ni]ST(\varphi))$

O. Gasquet and A. Herzig. *From Classical to Normal Modal Logic*. in Proof Theory of Modal Logic, Kluwer, pgs. 293 - 311, 1996.

M. Kracht and F. Wolter. *Normal Monomodal Logics can Simulate all Others*. The Journal of Symbolic Logic, 64:1, pgs. 99 - 138, 1999.

Richer Languages

- ✓ Normal + non-normal modalities
- First-order extensions
- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)

Neighborhood Models for First-Order Modal Logic

H. Arlo Costa and E. Pacuit. *First-Order Classical Modal Logic*. Studia Logica, **84**, pgs. 171 - 210 (2006).

Higher-Order Coalition Logic (time permitting)

G. Boella, D. Gabbay, V. Genovese, L. van der Torre. *Higher-Order Coalition Logic*. 2010.

First-Order Modal Language: $\mathcal{L}_1$

Extend the propositional modal language $\mathcal{L}$ with the usual first-order machinery (constants, terms, predicate symbols, quantifiers).

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$$A := P(t_1, \ldots, t_n) \mid \neg A \mid A \wedge A \mid \Box A \mid \forall x A$$

(note that equality is not in the language!)

State-of-the-art

T. Braüner and S. Ghilardi. *First-order Modal Logic*. Handbook of Modal Logic, pgs. 549 - 620 (2007).

D.Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*. Draft available (2008).

http://lpcs.math.msu.su/~shehtman/QNCLfinal.pdf

M. Fitting and R. Mendelsohn. *First-Order Modal Logic*. Kluwer Academic Publishers (1998).

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D\rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function I, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

A **substitution** is any function $\sigma : \mathcal{V} \to D$ ($\mathcal{V}$ the set of variables).

A substitution σ' is said to be an x**-variant** of σ if $\sigma(y) = \sigma'(y)$ for all variable y except possibly x, this will be denoted by $\sigma \sim_x \sigma'$.

First-order Modal Logic

A **constant domain Kripke frame** is a tuple $\langle W, R, D \rangle$ where W and D are sets, and $R \subseteq W \times W$.

A **constant domain Kripke model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $R(w) \subseteq [\![\varphi]\!]_{\mathcal{M},\sigma}$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

First-order Modal Logic

A **constant domain Neighborhood frame** is a tuple $\langle W, N, D\rangle$ where W and D are sets, and $N : W \to \wp(\wp(W))$.

A **constant domain Neighborhood model** adds a valuation function V, where for each n-ary relation symbol P and $w \in W$, $I(P, w) \subseteq D^n$.

Suppose that σ is a substitution.

1. $\mathcal{M}, w \models_\sigma P(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in I(P, w)$
2. $\mathcal{M}, w \models_\sigma \Box A$ iff $[\![\varphi]\!]_{\mathcal{M},\sigma} \in N(w)$
3. $\mathcal{M}, w \models_\sigma \forall x A$ iff for each x-variant σ', $\mathcal{M}, w \models_{\sigma'} A$

Example

Suppose that F is a unary predicate symbol, $\mathcal{V} = \{x, y\}$, and $\langle W, N, D, I \rangle$ is a first order constant domain neighborhood model where

- $W = \{w, v, u\}$;
- $N(w) = \{\{w, v\}, \{u\}\}$, $N(v) = \{\{v\}\}$, $N(u) = \{\{w, v\}, \{v\}\}$;
- $D = \{a, b\}$; and
- $I(F, w) = \{a\}$, $I(F, v) = \{a, b\}$, and $I(F, u) = \emptyset$.

Example

There are four possible substitutions:

- $\sigma_1 : \mathcal{V} \to D$ where $\sigma_1(x) = a$, $\sigma_1(y) = b$;
- $\sigma_2 : \mathcal{V} \to D$ where $\sigma_2(x) = b$, $\sigma_2(y) = a$;
- $\sigma_3 : \mathcal{V} \to D$ where $\sigma_3(x) = \sigma_3(y) = a$; and
- $\sigma_4 : \mathcal{V} \to D$ where $\sigma_4(x) = \sigma_4(y) = b$

- $[\![F(x)]\!]_{\mathcal{M},\sigma_1} = \{w, v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_2} = \{v\}$;
- $[\![F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, v\}$; and
- $[\![F(x)]\!]_{\mathcal{M},\sigma_4} = \{v\}$.

Example

In general, every formula $\varphi \in \mathcal{L}_1$ is associated with a function

$$[\![\varphi]\!] : D^{\mathcal{V}} \to \wp(W)$$

Example

- $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_1} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_3} = \{w, u\}$
  $[\![\Box F(x)]\!]_{\mathcal{M},\sigma_2} = [\![\Box F(x)]\!]_{\mathcal{M},\sigma_4} = \{v, u\}$;
- $[\![\Box \forall x F(x)]\!]_{\mathcal{M},\sigma_1} = \{v\}$; and
- $[\![\forall x \Box F(x)]\!]_{\mathcal{M},\sigma_1} = \{v, u\}$.

Barcan Schemas

- **Barcan formula** (BF): $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$
- **converse Barcan formula** (CBF): $\Box \forall x A(x) \rightarrow \forall x \Box A(x)$

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Observation 1: *CBF* is provable in **FOL** + **EM**

Observation 2: *BF* and *CBF* both valid on relational frames with constant domains

Observation 3: *BF* is valid in a *varying* domain relational frame iff the frame is anti-monotonic; *CBF* is valid in a *varying* domain relational frame iff the frame is monotonic.

See (Fitting and Mendelsohn, 1998) for an extended discussion

Constant Domains without the Barcan Formula

The system **EMN** and seems to play a central role in characterizing monadic operators of high probability (See Kyburg and Teng 2002, Arló-Costa 2004).

Constant Domains without the Barcan Formula

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Of course, *BF* should fail in this case, given that it instantiates cases of what is usually known as the ‘**lottery paradox**’:

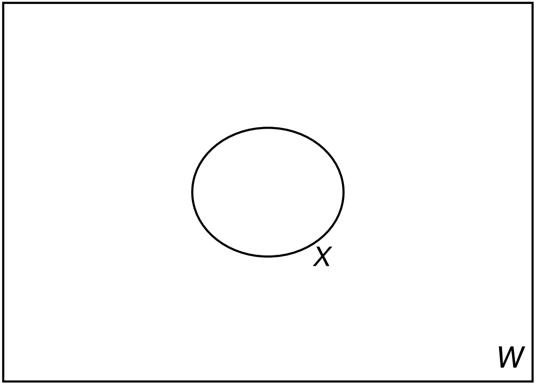
For each individual x, it is *highly probably* that x will loose the lottery; however it is not necessarily highly probably that each individual will loose the lottery.

Converse Barcan Formulas and Neighborhood Frames

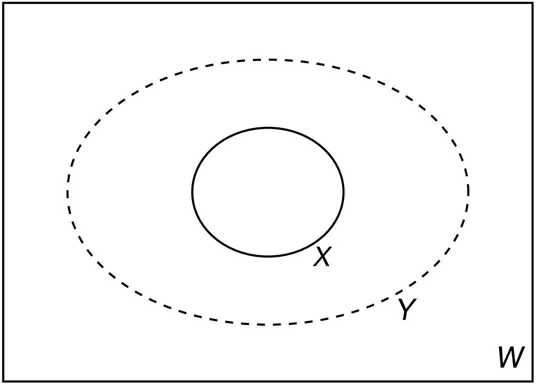
A frame $\mathcal{F}$ is **consistent** iff for each $w \in W$, $N(w) \neq \emptyset$

A first-order neighborhood frame $\mathcal{F} = \langle W, N, D \rangle$ is **nontrivial** iff $|D| > 1$

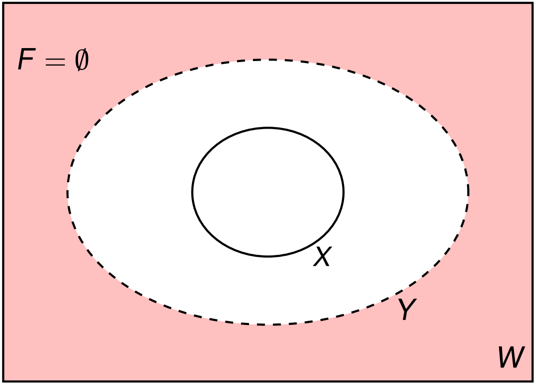
Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The converse Barcan formula is valid on $\mathcal{F}$ iff either $\mathcal{F}$ is trivial or $\mathcal{F}$ is supplemented.



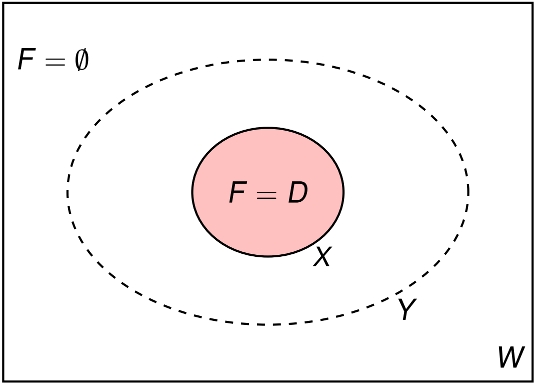
$X \in N(w)$



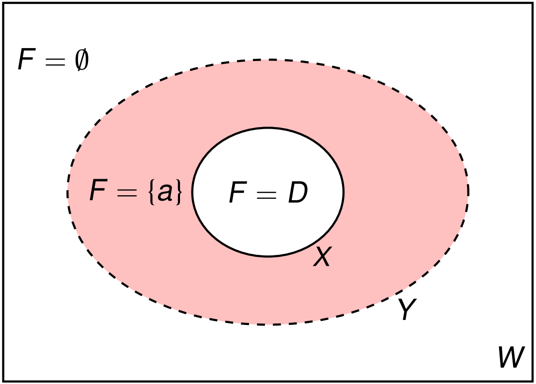
$Y \notin N(w)$



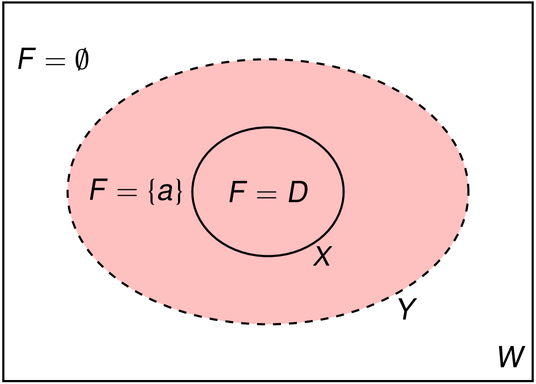
$$\forall v \notin Y, \;\; I(F, v) = \emptyset$$



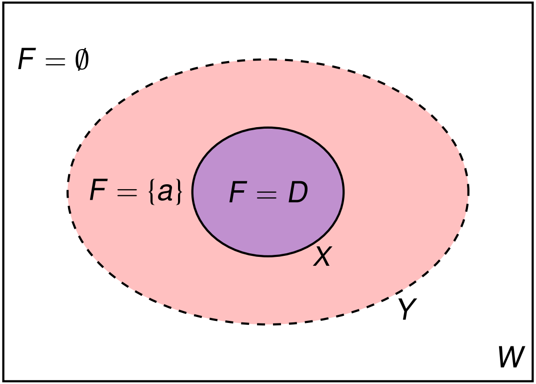
$$\forall v \in X, \;\; I(F, v) = D = \{a, b\}$$



$$\forall v \in Y - X, \;\; I(F, v) = D = \{a\}$$



$(F[a])^{\mathcal{M}} = Y \notin N(w)$ hence $w \not\models \forall x \Box F(x)$



$$(\forall x F(x))^{\mathcal{M}} = (F[a])^{\mathcal{M}} \cap (F[b])^{\mathcal{M}} = X \in N(w)$$

$$\text{hence} \quad w \models \Box \forall x F(x)$$

Barcan Formulas and Neighborhood Frames

We say that a frame closed under $\leq \kappa$ intersections if for each state w and each collection of sets $\{X_i \mid i \in I\}$ where $|I| \leq \kappa$, $\cap_{i \in I} X_i \in N(w)$.

Lemma Let $\mathcal{F}$ be a consistent constant domain neighborhood frame. The Barcan formula is valid on $\mathcal{F}$ iff either

1. $\mathcal{F}$ is trivial or
2. if D is finite, then $\mathcal{F}$ is closed under finite intersections and if D is infinite and of cardinality κ, then $\mathcal{F}$ is closed under $\leq \kappa$ intersections.

Suppose that **L** is a propositional modal logic. Let **FOL** $+$ **L** denote the set of formulas closed under the following rules and axiom schemes

L All axiom schemes and rules from **L**.

(All) $\forall x\varphi(x) \rightarrow \varphi[y/x]$ is an axiom scheme, where y is free for x in φ.

(Gen) $\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x\psi}$, where x is not free in φ.

Theorem FOL $+$ E is sound and strongly complete with respect to the class of **all** constant domain neighborhood frames.

CBF

$$\vdash_{\textbf{FOL+EM}} \Box\forall x\varphi(x) \rightarrow \forall x\Box\varphi(x)$$

$$\nvdash_{\textbf{FOL+E}+\text{(CBF)}} \Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$$

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Theorem FOL $+$ **E** $+$ *CBF* is sound and strongly complete with respect to the class of frames that are either non-trivial and supplemented or trivial and not supplemented.

FOL $+$ **K** and **FOL** $+$ **K** $+$ *BF*

Theorem **FOL** $+$ **K** is sound and strongly complete with respect to the class of filters.

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Observation The augmentation of the smallest canonical model for **FOL** $+$ **K** is not a canonical model for **FOL** $+$ **K**. In fact, the closure under infinite intersection of the minimal canonical model for **FOL** $+$ **K** is not a canonical model for **FOL** $+$ **K**.

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Lemma The augmentation of the smallest canonical model for **FOL** $+$ **K** $+$ *BF* is a canonical for **FOL** $+$ **K** $+$ *BF*.

Theorem **FOL** $+$ **K** $+$ *BF* is sound and strongly complete with respect to the class of augmented first-order neighborhood frames.

Richer Languages

- Normal + non-normal modalities
- First-order extensions
- Fixed-point operators/group notions (group evidence, common belief)
- Dynamic extensions (game logic, updating neighborhood models, evidence dynamics)

Background: Modeling Informational Changes

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Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model and $\varphi \in \mathcal{L}$ a formula. The model $\mathcal{M}^{!\varphi} = \langle W^{!\varphi}, E^{!\varphi}, V^{!\varphi} \rangle$ is defined as follows: $W^{!\varphi} = [\![\varphi]\!]_{\mathcal{M}}$, for each $p \in \mathsf{At}$, $V^{!\varphi}(p) = V(p) \cap W^{!\varphi}$ and for all $w \in W$,

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$[!\varphi]\psi$: “ψ is true after the public announcement of φ”

$\mathcal{M}, w \models [!\varphi]\psi$ iff $\mathcal{M}, w \models \varphi$ implies $\mathcal{M}^{!\varphi}, w \models \psi$

Public Announcements: Recursion Axioms

$$
\begin{array}{lll}
[!\varphi]p & \leftrightarrow & (\varphi \rightarrow p) \qquad (p \in \mathrm{At}) \\
[!\varphi](\psi \wedge \chi) & \leftrightarrow & ([!\varphi]\psi \wedge [!\varphi]\chi) \\
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[!\varphi]B\psi & \leftrightarrow & (\varphi \rightarrow B^{\varphi}[!\varphi]\psi) \\
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1. Other definition of public announcement
2. Dissecting the public announcement operation

Public Announcement

Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is a monotonic neighborhood modeland $\emptyset \neq X \subseteq W$.

Intersection submodel
$N^{\Cap X}(w) = \{Y \mid \emptyset \neq Y = X \cap Z \text{ for some } Z \in N(w)\}$

Strong intersection submodel:
$N^{\cap X}(w) = \{Y \mid Y = Z \cap X \text{ for some } Z \in N(w)\}$.

Subset submodel: $N^{\subseteq X}(w) = \{Y \mid Y \subseteq X \text{ and } Y \in N(w)\}$.

- $[\varphi]^{\cap}\Box\psi \leftrightarrow (\varphi \rightarrow \Box[\varphi]^{\cap}\psi)$ is **valid** on monotonic frames.

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- Suppose that $\mathcal{M} = \langle W, N, V \rangle$ is augmented. Then, for any formula φ, $\mathcal{M}^{\cap\varphi} = \mathcal{M}^{\subseteq\varphi}$.

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Evidence Addition

Let $\mathcal{M} = \langle W, E, V \rangle$ be an evidence model, and φ a formula in $\mathcal{L}$. The model $\mathcal{M}^{+\varphi} = \langle W^{+\varphi}, E^{+\varphi}, V^{+\varphi} \rangle$ has $W^{+\varphi} = W$, $V^{+\varphi} = V$ and for all $w \in W$,

$$E^{+\varphi}(w) = E(w) \cup \{[\![\varphi]\!]_{\mathcal{M}}\}$$

$[+\varphi]\psi$: "ψ is true after φ is accepted as an admissible piece of evidence"

$\mathcal{M}, w \models [+\varphi]\psi$ iff $\mathcal{M}, w \models E\varphi$ implies $\mathcal{M}^{+\varphi}, w \models \psi$

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\end{array}$$

Evidence Addition: Recursion Axioms

$$[+\varphi]\Box\psi \quad\leftrightarrow\quad (E\varphi \to (\Box[+\varphi]\psi \vee A(\varphi \to [+\varphi]\psi)))$$

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Evidence Addition: Recursion Axioms

$$[+\varphi]B\psi \quad \leftrightarrow \quad ????$$

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Adding φ

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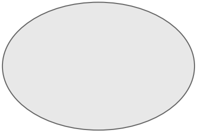
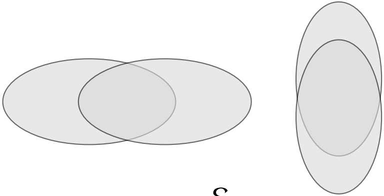
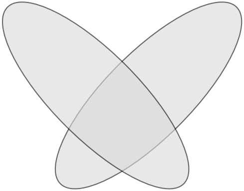


$\mathcal{E}_1$

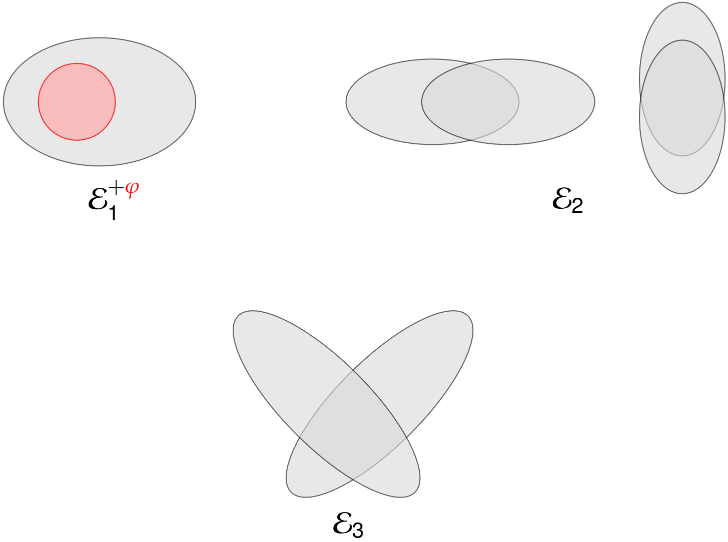


$\mathcal{E}_2$

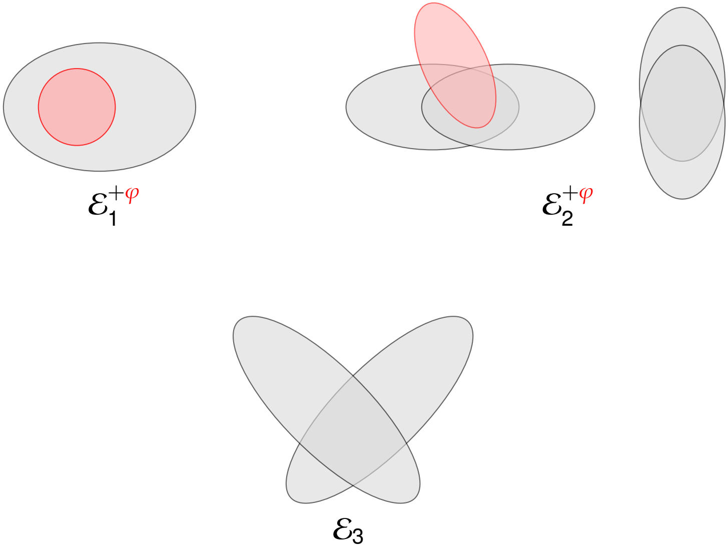


$\mathcal{E}_3$

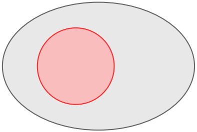
Adding φ



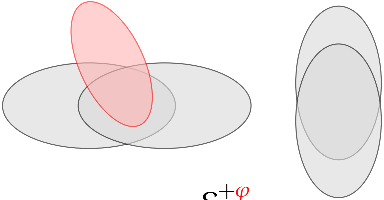
Adding φ



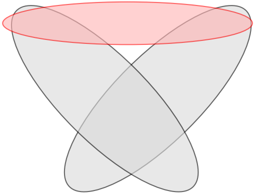
Adding φ



$\mathcal{E}_1^{+\varphi}$



$\mathcal{E}_2^{+\varphi}$



$\mathcal{E}_3^{+\varphi}$

Compatibile vs. Incompatible

Compatibile vs. Incompatible

1. $\mathcal{X}$ is maximally φ**-compatible** provided $\cap\mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \neq \emptyset$ and no proper extension $\mathcal{X}'$ of $\mathcal{X}$ has this property; and

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Conditional belief: $B^{+\varphi}\psi$ iff for each maximally φ-compatible $\mathcal{X} \subseteq E(w)$, $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$

Conditional Beliefs (Incompatibility Version): $\mathcal{M}, w \models B^{-\varphi}\psi$ iff for all maximal f.i.p., if $\mathcal{X}$ is incompatible with φ then $\bigcap \mathcal{X} \subseteq [\![\psi]\!]_{\mathcal{M}}$.

$B^{+\neg\varphi}$ vs. $B^{-\varphi}$

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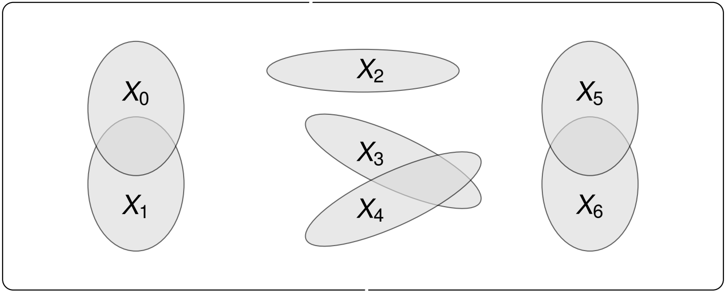

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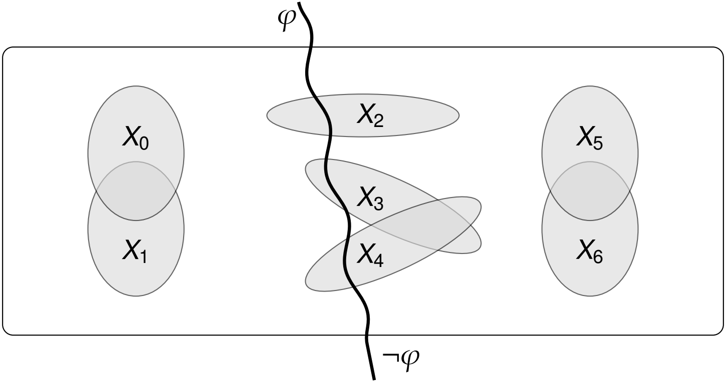
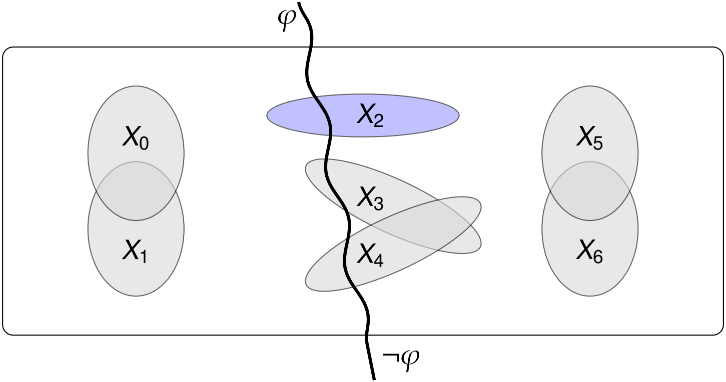


$B^{+\neg\varphi}$ vs. $B^{-\varphi}$



$\{X_2\}$ is (max.) compatible with $\neg\varphi$ but not maximally φ incompatible

Recursion Axiom

Fact. $[+\varphi]B\psi \leftrightarrow (E\varphi \rightarrow (B^{+\varphi}[+\varphi]\psi \wedge B^{-\varphi}[+\varphi]\psi))$ is valid.

Proof Sketch

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Language Extension: $\mathcal{M}, w \models B^{\varphi,\psi}\chi$ iff for all maximally φ-compatible sets $\mathcal{X} \subseteq E(w)$, if $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\psi]\!]_{\mathcal{M}}$, then $\bigcap \mathcal{X} \cap [\![\varphi]\!]_{\mathcal{M}} \subseteq [\![\chi]\!]_{\mathcal{M}}$.

$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

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$B^{+\varphi}$ is $B^{\varphi,\top}$ and $B^{-\varphi}$ is $B^{\top,\neg\varphi}$

Fact. The following is valid:

$$[+\varphi]B^{\psi,\alpha}\chi \leftrightarrow (E\varphi \rightarrow (B^{\varphi\wedge[+\varphi]\psi,[+\varphi]\alpha}[+\varphi]\chi \wedge B^{[+\varphi]\psi,\neg\varphi\wedge[+\varphi]\alpha}[+\varphi]\chi))$$

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Evidence Management

Evidence Removal: $E^{-\varphi}(w) = E(w) - \{X \mid X \subseteq [\![\varphi]\!]_{\mathcal{M}}\}$

$\mathcal{M}, w \models [-\varphi]\psi$ iff $\mathcal{M}, w \models \neg A\varphi$ implies $\mathcal{M}^{-\varphi}, w \models \psi$

Evidence Management

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- Are $\neg[\#]\Box\neg\varphi \rightarrow B\varphi$ and $[\#]\Box\varphi \rightarrow B\varphi$ valid?

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One-round evidence combination:

$$E^{\#_1}(w) = E(w) \cup \{X \mid \text{there are } Y_1, Y_2 \in E(w) \text{ with } \emptyset \neq X = Y_1 \cap Y_2\}$$

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Fact. $(E(\varphi \wedge \psi) \wedge \boxplus\varphi \wedge \boxplus\psi) \rightarrow [\#_1] \boxplus (\varphi \wedge \psi)$. is valid.

Evidence Combination (2)

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$[\#]\varphi$: "φ is true after the agent (consistently) combines (*all* of) her evidence"

$\mathcal{M}, w \models [\#]\varphi$ iff $\mathcal{M}^{\#}, w \models \varphi$.

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3. $B\varphi \rightarrow [\#]\Box\varphi$ (beliefs are explicitly supported after consistently combining evidence)

4. For factual φ, $B\varphi \rightarrow \neg[\#]\Box\neg\varphi$ (if an agent believes φ then the agent cannot combine her evidence so that there is evidence for $\neg\varphi$)

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$B\varphi \to \Box\varphi$ vs. $B\varphi \to [\#]\Box\varphi$

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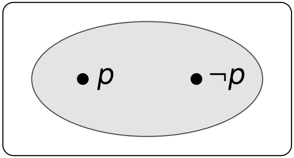
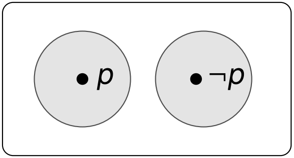
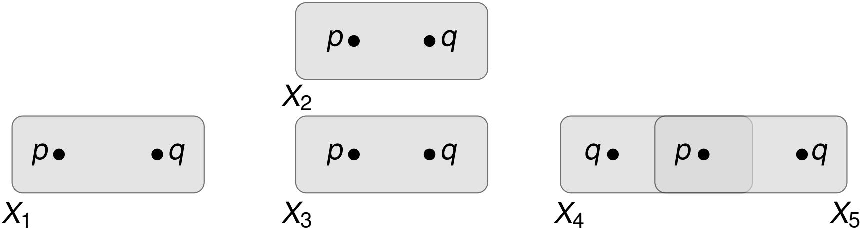


$\mathcal{E}_1$



$\mathcal{E}_2$

Different Evidential Situations



$\mathcal{E}_1 = \{X_1\}$ $\mathcal{E}_2 = \{X_2, X_3\}$ $\mathcal{E}_3 = \{X_4, X_5\}$

Richer Languages

- ✓ Normal + non-normal modalities
- ✓ First-order extensions
- Dynamics
  - ✓ Dynamics on neighborhoods (updating neighborhood models, evidence dynamics)
  - Dynamics with neighborhoods (Game logic)
- Fixed-point operators/group notions (group evidence, common belief)

Neighborhood Semantics for Modal Logic

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